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Hamiltonian systems on k-cosymplectic manifolds

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The Hamiltonian framework on symplectic and cosymplectic manifolds is extended in order to consider classical field theories. To do this, the notion of k-cosymplectic manifold is introduced, and a suitable Hamiltonian formalism is developed so that the field equations for scalar and vector Hamiltonian functions are derived. © *1998 American Institute of Physics.* [S0022-2488(98)01102-5]

I. INTRODUCTION

There are many attempts in recent years to develop a Hamiltonian formalism for classical field theories. In the 1960s the so-called multisymplectic formalism was developed by the Tulczyjew school in Warsaw (see Refs. 1–4) and independently by García and Pérez-Rendón[5,6] and Goldschmidt and Sternberg.[7] This approach was revisited by Martin,[8,9] Gotay,[10,11] and more recently by Cantrijn *et al.*,[12–14] de León *et al.*,[15,16] Sardanashvily,[17,18] Kanatchikov,[19] and Echevarría *et al.*[20] (also see Refs. 21 and 22).

At the same time, other approaches were suggested, with different names but defining the same geometrical object. Thus, Gunther introduced the so-called polysymplectic structures,[23] Awane[24,25] the k-symplectic manifolds, also discussed by Puta[26] and Norris (see Ref. 27 and the references therein), and de León *et al.*[28,29] studied the so-called p-almost cotangent structures. Along this paper, we shall adopt the name of a (almost) k-symplectic structure. The original idea in all these papers was to abstract the geometrical ingredients of the k-cotangent bundle $(T_k^1)^*Q$ of a manifold Q, say the manifold of 1-jets with target $0 \in \mathbf{R}^k$ of all the mappings from Q into $\mathbf{R}^k$. Since $(T_k^1)^*Q$ may be canonically identified with the Whitney sum of k copies of T^*Q, we can transport the canonical symplectic structure on it to $(T_k^1)^*Q$. It should be noticed that the vertical distribution has to be incorporated in the game. This structure can be also derived by using the solder form of the linear frame bundle of Q (see Ref. 27). We also remark that in Ref. 29 it was presented an integrability theorem that provides Darboux coordinates as in the symplectic case.

Given a k-symplectic manifold, we obtain a Hamiltonian system by introducing a Hamiltonian function. Thus, the corresponding field equations can be derived[23] as the integral sections of a k-vector field. Moreover, if we consider vector-valued Hamiltonian functions, we obtain a Hamiltonian vector field, and in this case, the Hamiltonian has to be of polynomial form.[24,25]

The purpose of this paper is to extend the study to the case of field equations involving the independent parameters in an explicit way; in other words, the Hamiltonian function is of the form $H(t^i, x^\alpha, x_\alpha^i)$, where $1 \leq i \leq k$, $1 \leq \alpha \leq n$. To define a convenient geometrical structure, we extract the geometric ingredients of the so-called stable cotangent bundle of k^1-covelocities $\mathbf{R}^k$

$\times(T_k^1)^*Q$ of a manifold Q. Thus, a family $(\eta_i,\omega_i,V;\ 1\leqslant i\leqslant k)$, where each η_i is a 1-form, each ω_i is a 2-form, and V is an nk-dimensional distribution on a $k(n+1)+n$-dimensional manifold M satisfying some compatibility relations, will be called an almost k-cosymplectic structure on M, and M an almost k-cosymplectic manifold. After identifying such a structure as a kind of G-structure on M, an integrability theorem is proved and Darboux coordinates are introduced on a k-cosymplectic manifold (Sec. II). This result shows that a k-cosymplectic manifold is locally isomorphic to a stable cotangent bundle of k^1-covelocities; in Sec. III we prove that, under some global conditions, there is also a global isomorphism. In Sec. IV, we introduce the dynamics; in fact, the geometry of a k-cosymplectic manifold allows us to write the field equations in a global way. An example is discussed. The case of vector-valued Hamiltonian functions is studied in Sec. V. Finally, we compare our geometrical approach with the so-called multisymplectic formalism (Sec. VI). In a forthcoming paper we shall examine the corresponding Euler–Lagrange formalism.

II. ALMOST *k*-COSYMPLECTIC STRUCTURES

Let $\pi:\mathbf{R}^k\times Q\to Q$ be a trivial fibered manifold and denote by $J^1\pi$ the manifold of 1-jets of local sections of π, $J^1\pi$ will be called the stable cotangent bundle of k^1-covelocities of the manifold Q. $J^1\pi\to Q$ is a vector bundle over Q with standard fiber $\mathbf{R}^k\times\mathbf{R}^{nk}$, where $\dim Q=n$. We have a canonical identification $J^1\pi\cong\mathbf{R}^k\times(T_k^1)^*Q$ in such a way that we can introduce local coordinates on $\mathbf{R}^k\times(T_k^1)^*Q$ as follows. If (x^α) are local coordinates on Q, then $(t^i,x^\alpha,x_\alpha^i)$, $1\leqslant i\leqslant k$, $1\leqslant\alpha\leqslant n$, is a local coordinate system on $\mathbf{R}^k\times(T_k^1)^*Q$. The reason for the above notation is the following. For $k=1$ we have $J^1\pi\cong\mathbf{R}\times T^*Q$, and (t,q^α,p_α) stands for the canonical coordinates on it.

Define a family $((\eta_0)_i,(\omega_0)_i,V_0)$ of 1-forms $(\eta_0)_i$, 2-forms $(\omega_0)_i$ and a distribution V_0 given by

$$(\eta_0)_i=dt^i,\quad (\omega_0)_i=dx^\alpha\wedge dx_\alpha^i,\quad V_0=\left\langle\frac{\partial}{\partial x_\alpha^1},\dots,\frac{\partial}{\partial x_\alpha^k}\right\rangle.$$

A simple inspection in local coordinates shows that the following relations hold:

(i) $(\eta_0)_1\wedge\cdots\wedge(\eta_0)_k\neq 0$,
(ii) $\dim(\ker(\omega_0)_1\cap\cdots\cap\ker(\omega_0)_k)=k$,
(iii) $\ker(\eta_0)_1\cap\cdots\cap\ker(\eta_0)_k\cap\ker(\omega_0)_1\cap\cdots\cap\ker(\omega_0)_k=\{0\}$,
(iv) $(\eta_0)_{i_{\rfloor V0}}=0,\quad(\omega_0)_{i_{\rfloor V0\times V0}}=0\quad(1\leqslant i\leqslant k)$.

Inspired in the above geometrical model, we introduce the following definition.

Definition II.1: Let M be a differentiable manifold of dimension $k(n+1)+n$. A family $(\eta_i,\omega_i,V;\ 1\leqslant i\leqslant k)$ where each η_i is a 1-form, each ω_i is a 2-form, and V is an nk-dimensional distribution on M, such that

(i) $\eta_1\wedge\cdots\wedge\eta_k\neq 0$,
(ii) $\dim(\ker\,\omega_1\cap\cdots\cap\ker\,\omega_k)=k$,
(iii) $\ker\,\eta_1\cap\cdots\cap\ker\,\eta_k\cap\ker\,\omega_1\cap\cdots\cap\ker\,\omega_k=\{0\}$,
(iv) $\eta_{i_{\rfloor V}}=0,\quad\omega_{i_{\rfloor V\times V}}=0\quad(1\leqslant i\leqslant k)$,

will be called an *almost k-cosymplectic structure*, and the manifold M an *almost k-cosymplectic manifold.*

In particular, if $k=1$, then $\dim M=2n+1$, and (η,ω) is an almost cosymplectic structure on M,[30,31] and (η,ω,V) an almost stable cotangent structure on M.[32] From the conditions of Definition II.1 we deduce that there exist k vector fields $\xi_1,\dots,\xi_k$ on M satisfying

$$\iota_{\zeta_i}\eta_j=\delta_{ij},\quad\iota_{\xi_i}\omega_j=0,\tag{1}$$

with $1\leqslant i$, $j\leqslant k$, and that will be called the Reeb vector fields associated to the almost k-cosymplectic structure. Our intention is to get integrability conditions that provide Darboux coordinates for such a kind of geometrical structure. To go further in this direction, we first shall describe an almost k-cosymplectic structure as a type of G-structure. Let us recall that a G-structure on an m-dimensional manifold M is a G-reduction of the linear frame bundle of M.[33] Here G a Lie subgroup of $Gl(m,\mathbf{R})$.

Define k distributions $V_1,\ldots,V_k$ on M by

$$V_i=\cap_{j\neq i}\ker\ \omega_j\cap\ker\ \eta_1\cap\cdots\cap\ker\ \eta_k .$$

Then

$$V_i\subset V,\quad \dim V_i=n,\quad V_i\cap(+_{j\neq i}V_j)=\{0\},$$

for all $1\leqslant i\leqslant k$, and therefore we have

$$V=V_1\oplus\cdots\oplus V_k .$$

Since (1) we can choose a linear frame at a point $x\in M$ of the form $\{\xi_1,\ldots,\xi_k,Y_{k+1},\ldots,Y_{k(n+1)+n}\}$, such that

$$\langle Y_{k+1},\ldots,Y_{k(n+1)+n}\rangle=\ker\ \eta_1\cap\cdots\cap\ker\ \eta_k ,$$

$$\langle Y_{k+in+1},\ldots,Y_{k+in+n}\rangle=V_i ,$$

for all $1\leqslant i\leqslant k$. Moreover, we have $\langle\xi_1,\ldots,\xi_k\rangle=\ker\ \omega_1\cap\cdots\cap\ker\ \omega_k$. With respect to this linear frame, each ω_i is written as

$$\omega_i=\begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0\\ 0 & P_i & \cdots & -Q_i^t & \cdots & 0\\ \vdots & \vdots & & \vdots & & \vdots\\ 0 & Q_i & \cdots & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots & & \vdots\\ 0 & 0 & \cdots & 0 & \cdots & 0\end{pmatrix},$$

with $P_i=-P_i^t$ and $\det Q_i\neq 0$. We construct a new frame $\{\xi_1,\ldots,\xi_k,X_{k+1},\ldots,X_{k(n+1)+n}\}$ at x given by the formula

$$\begin{pmatrix}\xi_1\\ \vdots\\ \xi_k\\ X_{k+1}\\ \vdots\\ X_{k(n+1)+n}\end{pmatrix}=(\xi_1,\ldots,\xi_k,Y_{k+1},\ldots,Y_{k(n+1)+n})\begin{pmatrix} I_k & 0 & 0 & \cdots & 0\\ 0 & A & 0 & \cdots & 0\\ 0 & B_1 & C_1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & B_k & 0 & \cdots & C_k\end{pmatrix},$$

where

$$A=Q_k^{-1},\quad C_k=I_n,\quad B_k=\tfrac{1}{2}\,(Q_k^{-1})^tP_kQ_k^{-1},$$

$$C_i=(Q_i^{-1})^tQ_k^t,\quad B_i=\tfrac{1}{2}\,(Q_i^{-1})^tP_iQ_k^{-1},$$

for all $1\leqslant i\leqslant k$. With respect to this new frame (called adapted to the k-cosymplectic structure) the forms $\omega_1,\ldots,\omega_k$, are expressed by the matrices $(\omega_1)_0,\ldots,(\omega_k)_0$ given by

$$(\omega_1)_0=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & -I_n & \cdots & 0\\ 0 & I_n & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix},\ldots,(\omega_k)_0=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & -I_n\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & I_n & 0 & \cdots & 0\end{pmatrix}.$$

Each pair of adapted frames is related by a change with associated matrix

$$\begin{pmatrix} I_k & 0 & 0 & \cdots & 0\\ 0 & A & 0 & \cdots & 0\\ 0 & B_1 & C & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & B_k & 0 & \cdots & C\end{pmatrix},$$

with $B_i^tA=A^tB_i$ and $C=(A^{-1})^t$ for all $1\leqslant i\leqslant k$. The group G of such matrices is a Lie subgroup of $Gl(k(n+1)+n,\mathbf{R})$. Now we obtain a G-structure on M by collecting all the adapted linear frames at all the points of M. Conversely, let us suppose a given G-structure $\mathbf{B}$ on M. Then we may define an almost k-cosymplectic structure $(\eta_i,\omega_i,V;1\leqslant i\leqslant k)$ on M, as follows. If $\{e_1,\ldots,e_{k(n+1)+n}\}$ is the canonical basis of $\mathbf{R}^{k(n+1)+n}$ and $\{e^1,\ldots,e^{k(n+1)+n}\}$ is the corresponding dual basis, we set $(\eta_i)_x(X)=e^i(u^{-1}(X))$, $(\omega_i)_x(X,Y)=(u^{-1}(X)(\omega_i)_0(u^{-1}(Y))^t$, and

$$V_x=u(\langle e_{k+n+1},\ldots,e_{k(n+1)+n}\rangle),$$

where $x\in M$, $X,Y\in T_xM$, and $u\in B$ is an adapted linear frame,

$$u:\mathbf{R}^{k(n+1)+n}\rightarrow T_xM.$$

Summing up, we have proved the following.

Proposition II.2: A manifold M of dimension $k(n+1)+n$ admits an almost k-cosymplectic structure if and only if the structure group of its tangent bundle is reducible to the group G of matrices of the form

$$\begin{pmatrix} I_k & 0 & 0 & \cdots & 0\\ 0 & A & 0 & \cdots & 0\\ 0 & B_1 & C & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & B_k & 0 & \cdots & C\end{pmatrix},$$

with $B_i^tA=A^tB_i$ and $C=(A^{-1})^t$ for all $1\leqslant i\leqslant k$.

Definition II.3: Let M be a $k(n+1)+n$-dimensional manifold with an almost k-cosymplectic structure (η_i,ω_i,V). We say that (η_i,ω_i,V) is integrable if the corresponding G-structure is integrable, and in such a case it is called *k-cosymplectic.*

A G structure on a manifold M is said to be integrable[33] if around each point of M there exist local coordinates (z^u) such that the local frame $\{\partial/\partial z^u\}$ is adapted to the G-structure, that is, it takes values in the reduced G-bundle.

Therefore, an almost k-cosymplectic structure (η_i,ω_i,V) on a manifold M is integrable if around each point of M there exist local coordinates $(s^i,x^\alpha,x^i_\alpha;1\leqslant i\leqslant k,1\leqslant\alpha\leqslant n)$ such that

$$\eta_i=ds^i,\quad \omega_i=dx^\alpha\wedge dx^i_\alpha,\quad V=\left\langle\frac{\partial}{\partial x^1_\alpha},\ldots,\frac{\partial}{\partial x^k_\alpha}\right\rangle_{\alpha=1,\ldots,n}. \tag{2}$$

Such coordinate functions will be called *Darboux* or *canonical coordinates*.

The following theorem characterizes the integrability of a k-cosymplectic structure.

Theorem II.4: *An almost k-cosymplectic structure* (η_i, ω_i, V) *on a manifold* M *is integrable if and only if the following conditions are satisfied:*

$$d\eta_i = 0, \quad d\omega_i = 0, \quad [V,V] \subset V \quad (1 \leqslant i \leqslant k). \tag{3}$$

Proof: The integrability of a G structure obviously implies (3). Conversely, let us suppose that (3) holds. Define $K_x = \ker \eta_1(x) \cap \cdots \cap \ker \eta_k(x)$, for each $x \in M$. We have

$$T_x M = \langle \xi_1 \rangle_x \oplus \cdots \oplus \langle \xi_k \rangle_x \oplus K_x,$$

and the distributions

$$\langle \xi_1 \rangle, \langle \xi_2 \rangle, \ldots, \langle \xi_k \rangle, K_x,$$

are all of them integrable. Moreover, the sum of each pair of these distributions is also integrable and $TM = \langle \xi_1 \rangle \oplus \cdots \oplus \langle \xi_k \rangle \oplus K$. From Lemma 1.3 of Ref. 34 we deduce that for every point $x \in M$ there exists a local coordinate system $(\overline{t_i}, \overline{x^\alpha}, \overline{x^i_\alpha}; 1 \leqslant i \leqslant k, 1 \leqslant \alpha \leqslant n)$ in a neighborhood $\overline{U}$ of x, such that

$$\langle \xi_i \rangle = \left\langle \frac{\partial}{\partial \overline{t_i}} \right\rangle, \quad K = \left\langle \frac{\partial}{\partial \overline{x^\alpha}}, \frac{\partial}{\partial \overline{x^i_\alpha}} \right\rangle_{\left(\substack{1 \leqslant \alpha \leqslant n \\ 1 \leqslant i \leqslant k}\right)}.$$

On $\overline{U}$, for each fixed $1 \leqslant i \leqslant k$, we have

$$\xi_i = f_i \frac{\partial}{\partial \overline{t_i}},$$

and, since $\eta_j(\xi_i) = \delta_{ij}$, we deduce that

$$\eta_i = \frac{1}{f_i} d\overline{t_i} = g_i \, d\overline{t_i}.$$

Since the 1-forms $\eta_1, \ldots, \eta_k$ are closed, we get

$$0 = d\eta_i = \frac{\partial g_i}{\partial \overline{t_j}} d\overline{t_j} \wedge d\overline{t_i} + \frac{\partial g_i}{\partial \overline{x^\alpha}} d\overline{x^\alpha} \wedge d\overline{t_i} + \frac{\partial g_i}{\partial \overline{x^\alpha_j}} d\overline{x^\alpha_j} \wedge d\overline{t_i},$$

which implies

$$\frac{\partial g_i}{\partial \overline{x^\alpha}} = 0 = \frac{\partial g_i}{\partial \overline{x^\alpha_j}},$$

for all $1 \leqslant i, j \leqslant k$, $1 \leqslant \alpha \leqslant n$, and

$$\frac{\partial g_i}{\partial \overline{t_j}} = 0,$$

for all $1 \leqslant i, j \leqslant k$, $i \neq j$. Then $g_i = g_i(\overline{t_i})$ and we can define k functions $h^i = h^i(\overline{t_i})$ where each h^i is a primitive function of g_i. Define a new coordinate system $(\widetilde{t_i}, \widetilde{x}^\alpha, \widetilde{x}^i_\alpha)$ on a neighborhood $\widetilde{U}$ of x by

$$\widetilde{t_i} = h^i(\overline{t_i}), \quad \widetilde{x}^\alpha = \overline{x^\alpha}, \quad \widetilde{x}^i_\alpha = \overline{x^i_\alpha},$$

for $1 \leqslant i \leqslant k$ and $1 \leqslant \alpha \leqslant n$. With respect to these coordinates, we write

$$\eta_i = d\tilde{t}_i, \quad \xi_i = \frac{\partial}{\partial \tilde{t}_i}, \quad K = \left\langle \frac{\partial}{\partial \tilde{x}^\alpha}, \frac{\partial}{\partial \tilde{x}^i_\alpha} \right\rangle_{\binom{1\leq\alpha\leq n}{1\leq i\leq k}}.$$

Since K is an integrable distribution there exists an integral submanifold W of K through x such that $(W, \omega_1, \ldots, \omega_k, V)$ is a k-symplectic manifold (see Refs. 24, 25, 28, and 29). Then there exists a coordinate neighborhood $(U', x^\alpha, x^i_\alpha)$ with $1 \leq i \leq k$, $1 \leq \alpha \leq n$, in W such that

$$\omega_{i_{]U'}} = dx^\alpha \wedge dx^i_\alpha, \quad V_{]U'} = \left\langle \frac{\partial}{\partial x^1_\alpha}, \ldots, \frac{\partial}{\partial x^k_\alpha} \right\rangle_{1\leq\alpha\leq n},$$

for all $1 \leq i \leq k$. If the coordinate functions $\tilde{t}_i$ of $\tilde{U}$ are defined in $(-\epsilon_i, \epsilon_i)$, for each $1 \leq i \leq k$, we take the following coordinate neighborhood of x in M:

$$(\mathbf{U} = (-\epsilon_1, \epsilon_1) \times \cdots \times (-\epsilon_k, \epsilon_k) \times U', s^i, x^\alpha, x^i_\alpha),$$

with $s^i = \tilde{t}_i$, and then we have

$$\begin{aligned}\omega_i = {}& (A_i)^j_r\, ds^j \wedge ds^r + (B_i)^j_\alpha\, ds^j \wedge dx^\alpha + (C_i)^j_{\alpha,s} ds^j \wedge dx^r_\alpha \\ &+ (D_i)^\alpha_{\beta,j}\, dx^\alpha \wedge dx^j_\beta + (E_i)^\alpha_\beta\, dx^\alpha \wedge dx^\beta + (F_i)^{\alpha,j}_{\beta,s}\, dx^j_\alpha \wedge dx^r_\beta.\end{aligned}$$

Since $\iota_{\xi_j}\omega_i = 0$, we deduce that

$$(A_i)^j_r = (B_i)^j_\alpha = (C_i)^j_{\alpha,s} = 0,$$

for all $1 \leq i,j,\ s \leq k$, $1 \leq \alpha \leq n$, and from $d\omega_i = 0$ we obtain

$$\frac{\partial (E_i)^\alpha_\beta}{\partial s^j} = \frac{\partial (F_i)^{\alpha,j}_{\beta,s}}{\partial s^l} = \frac{\partial (D_i)^\alpha_{\beta,j}}{\partial s^r} = 0.$$

Therefore

$$(E_i)^\alpha_\beta(s^j, x^\sigma, x^j_\sigma) = (E_i)^\alpha_\beta(0, x^\sigma, x^j_\sigma) = 0,$$

$$(F_i)^{\alpha,j}_{\beta,s}(s^l, x^\sigma, x^l_\sigma) = (F_i)^{\alpha,j}_{\beta,s}(0, x^\sigma, x^l_\sigma) = 0,$$

$$(D_i)^\alpha_{\beta,j}(s^r, x^\sigma, x^r_\sigma) = (D_i)^\alpha_{\beta,j}(0, x^\sigma, x^r_\sigma) = \delta_{\alpha\beta}\delta_{ij},$$

from which we get that

$$\omega_i = dx^\alpha \wedge dx^i_\alpha.$$

Moreover, for each $x \in U$, we have

$$V(x) \subset T_x U',$$

and then

$$V = \left\langle \frac{\partial}{\partial x^1_\alpha}, \ldots, \frac{\partial}{\partial x^k_\alpha} \right\rangle_{1\leq\alpha\leq n},$$

which completes the proof of the theorem. ■

Remark II.5: When $k = 1$, the condition $[\xi, V] \subset V$ is a necessary condition for the integrability of the almost stable cotangent structure (η, ω, V), and it is an independent condition of the other ones (see Ref. 32). For $k > 1$, we have the following.

Proposition II.6: If M is an almost k-cosymplectic manifold with $k > 1$, and $\xi_1, \ldots, \xi_k$ are the Reeb vector fields associated to the almost k-cosymplectic structure, then

$$(d\eta_j=0, d\omega_j=0, \quad \forall 1\leqslant j\leqslant k)\Rightarrow[\xi_i, V]\subset V,$$

for all $1\leqslant i\leqslant k$.

III. *k*-COSYMPLECTIC MANIFOLDS THAT ARE STABLE *k*-COTANGENT BUNDLES

Let $(M,\eta_i,\omega_i,V;1\leqslant i\leqslant k)$ be a k-cosymplectic manifold. For each $1\leqslant i\leqslant k$, we consider the distributions

$$V_i=\ker\ \eta_1\cap\cdots\cap\ker\ \eta_k\cap_{j\neq i}\ker\ w_j,$$

$$K_i=\ker\ \eta_1\cap\cdots\cap\ker\ \eta_k\cap\ker\ \omega_i=\oplus_{j\neq i}V_j,$$

which are all of them involutive and hence define foliations on M. If $\xi_1,\ldots,\xi_k$ are the Reeb vector fields, then we put

$$W=\langle\xi_1,\ldots,\xi_k\rangle\oplus V,$$

$$W_i=\langle\xi_1,\ldots,\hat{\xi}_i,\ldots,\xi_k\rangle\oplus K_i,$$

where ˆ over an element means that this element is omitted. Again, W and W_i are integrable distributions of dimension $k(n+1)$ and $(n+1)(k-1)$, respectively.

Definition III.1: The k-cosymplectic structure (η_i,ω_i,V) defines a fibration on M when the following conditions are satisfied.

(i) The leaf space $M_i=M/W_i$ defined by the involutive distribution W_i is a quotient manifold of M, and the canonical projection $p_i:M\to M_i$ is a fibration for each $1\leqslant i\leqslant k$.
(ii) The leaf space $Q=M/W$ defined by the involutive distribution W is a quotient manifold of M and the canonical projection $p:M\to Q$ is a fibration.

In such a case, we have the following commutative diagram:

$$\begin{tikzcd} M \arrow[rr, "p_i"] \arrow[dr, "p"'] & & M_i \arrow[dl, "\sigma_i"] \\ & Q & \end{tikzcd}$$

where σ_i is the induced projection. Moreover, we can choose a suitable coordinate system $(s^i,x^\alpha,x^i_\alpha)$, such that

$$p_i(s^1,\ldots,s^k,x^\alpha,x^1_\alpha\ldots,x^k_\alpha)=(s^i,x^\alpha,x^i_\alpha),$$

$$p(s^1,\ldots,s^k,x^\alpha,x^1_\alpha,\ldots,x^k_\alpha)=(x^\alpha),$$

$$\sigma_i(s^i,x^\alpha,x^i_\alpha)=(x^\alpha).$$

On each M_i we define a cosymplectic structure $(\overline{\eta_i},\overline{\omega_i})$ by putting

$$\overline{\eta_i}(X)=\eta_i(\overline{X}),\quad \overline{\omega_i}(X,Y)=\omega_i(\overline{X,Y}),$$

with $\overline{X,Y}\in TM$, such that $(p_i)_*(\overline{X})=X$ and $(p_i)_*(\overline{Y})=Y$. Moreover, each V_i defines a distribution $\overline{V_i}$ on M_i such that $(\overline{\eta_i},\overline{\omega_i},\overline{V_i})$ is a stable almost cotangent structure on M.[32]

Definition III.2: Let $y\in M$ be such that $p(y)=x$ and $p_i(y)=y_i$ for each $1\leqslant i\leqslant k$. If $\beta\in T^*_xQ$, the i-vertical lift of β to T_yM is the unique vector $\beta^{(i)}\in T_yM$, such that $\beta^{(i)}\in V_i(y)$ and

$$\iota_{T(p_i)\beta^{(i)}}\overline{\eta_i}=0,\quad \iota_{T(p_i)\beta^{(i)}}\overline{w_i}=(\sigma_i)^*\beta.$$

In a similar way to the case of k-symplectic and stable almost cotangent structures (see Ref. 29), we have the following results.

Proposition III.3: Given the 1-forms β, γ on Q, we have

$$[\beta^{(i)},\gamma^{(j)}]=0, \quad [\beta^{(i)},\xi_j]=0,$$

for all $1\leqslant i,\ j\leqslant k$.

Proposition III.4: If ∇ is a symmetric connectic adapted to the k-cosymplectic structure on M that define a fibration, then ∇ induces, by restriction, a flat connection on each leaf of the fibrations $p_i:M\to M_i$.

It should be noticed that such an adapted symmetric connection always exists from the general theory of G structures (see Ref. 33). Now we can prove the main theorem of this section.

Theorem III.5: *Let $(M,\eta_1,\ldots,\eta_k,\omega_1,\ldots,\omega_k,V)$ be a k-cosymplectic manifold that defines a fibration $p:M\to Q$. Let ∇ be a symmetric connection adapted to the structure. If the flat connection induced by ∇ is geodesically complete, each fiber of p is connected and simply connected and each leaf of $\langle\xi_i\rangle\oplus V_i$ is connected for all $1\leqslant i\leqslant k$, then M is an affine bundle modeled in the vector bundle $\mathbf{R}^k\times(T_k^1)^*Q$. Therefore, M admits a structure of vector bundle on Q isomorphic to $\mathbf{R}^k\times(T_k^1)^*Q$.*

Proof: We need only to define a mapping,

$$\rho:M\times_Q(\mathbf{R}^k\times(T_k^1)^*Q)\to M,$$

such that, for each $z\in Q$,

$$\rho_z:p^{-1}(z)\times(\tau^k)^{-1}(z)\to p^{-1}(z),$$

is a free and transitive action. For any $\beta\in T^*Q$ and any $t\in\mathbf{R}^k$, we can define k vector fields $\hat{\beta}_i^{t^k}=t^i\xi_i+\beta^{(i)}$. Each $(\hat{\beta}_i^{t^k})$ is a geodesic vector field and then its flow $\exp t\hat{\beta}_i^{t^i}$ is global. For any $(t^1,\ldots,t^k)\in\mathbf{R}^k$, $(\beta_1,\ldots,\beta_k)\in(T_k^1)^*Q$, and $y\in p^{-1}(z)$, we then define

$$\rho_z(y,t^1,\ldots,t^k,\beta_1,\ldots,\beta_k)=(\exp\hat{\beta}_k^{t^k})\cdots(\exp\hat{\beta}_2^{t^2})(\exp\hat{\beta}_1^{t^1})(y). \qquad \blacksquare$$

Therefore, M is an affine bundle modeled over $\mathbf{R}^k\times(T_k^1)^*Q$, and if $s:Q\to M$ is a global section of p we can define an isomorphism $F_s:M\to\mathbf{R}^k\times(T_k^1)^*Q$ by identifying s with the zero section in M.

Proposition III.6: For any global section s of p, the diffeomorphism $F_s:M\to\mathbf{R}^k\times(T_k^1)^*Q$ defined above, satisfies

$$\eta_i=F_s^*((\eta_0)_i+(\tau^k)^*\beta_i), \quad \omega_i=F_s^*((\omega_0)_i+(\tau^k)^*\phi_i),$$

where $(\eta_0)_i$, $(\omega_0)_i$ are the canonical forms on $\mathbf{R}^k\times(T_k^1)^*Q$.

Proof: The result follows from the formulas:

$$F_*(\alpha^{(i)})=\alpha^{(i)0}, \quad F_*(\xi_i)=(\xi_0)_i,$$

where $\alpha^{(i)}$ and $\alpha^{(i)_0}$ are the (i) lifts of α to M and $\mathbf{R}^k\times(T_k^1)^*Q$, respectively, and ξ_i, $(\xi_0)_i$ are the Reeb vector fields on M and $\mathbf{R}^k\times(T_k^1)^*Q$, respectively. $\blacksquare$

Definition III.7: A k-cosymplectic structure satisfying all the hypotheses of Theorem III.5 will be called *regular*.

Two regular k-cosymplectic structures $(M,p,Q,\eta_i\omega_i,V)$ and $(\overline{M},\overline{p},Q,\overline{\eta_i},\overline{\omega_i},\overline{V})$ over the same manifold Q are equivalent if there exists a bundle isomorphism $F:M\to M$ over the identity on Q, such that

$$F^*\overline{\eta_i}-\eta_i=p^*(df_i), \quad F^*\overline{\omega_i}-\omega_i=p^*(d\beta_i),$$

for all $1\leqslant i\leqslant k$, where f_i is a function on Q and β_i is a 1-form on Q. Then $F^*\overline{\eta_i}-\eta_i$ and $F^*\overline{\omega_i}-\omega_i$ are cohomologous to zero for all $1\leqslant i\leqslant k$.

Proposition III.8: There is a 1–1 correspondence between the set of equivalence classes of regular k-cosymplectic structures over Q and the elements of

$$(H^1(Q,\mathbf{R})\times\cdots\times H^1(Q,\mathbf{R}))\oplus(H^2(Q,\mathbf{R})\times\cdots\times H^2(Q,\mathbf{R})),$$

where $H^1(Q,R)$ *and* $H^2(Q,R)$ *are, respectively, the first and the second de Rham cohomology groups of* Q.

The above results are a natural extension of those obtained by Thompson and Schwardmann.[35,36]

An alternative procedure can be developed by using the following theorem due to Nagano.[37]

Theorem III.9: *Suppose that there exists a vector field C on a manifold M satisfying the following conditions.*

- (i) *C generates a global one-parameter transformation group on M.*
- (ii) *For each point $x\in M$ there exists a unique* $\lim_{t\to-\infty}(\exp tC)(x)$, where $\exp tC$ *denotes the flow of C.*
- (iii) *The characteristic operator $(A_C)_x$ associated to C satisfies $((A_C)_x)^2=(A_C)_x$ for each singular point x of C.*
- (iv) *The set of S of the singular points of C is a submanifold of M such that codimension $S=\mathrm{rank}(A_C)_x$, for all $x\in S$.*

Then there exists a unique vector bundle structure on M such that C is the canonical vector field.

Theorem III.10: *Let M be a $k(n+1)+n$-dimensional manifold endowed with a k-cosymplectic structure $(\eta_i,\omega_i,V;1\leqslant i\leqslant k)$ such that the 1-forms η_i and the 2-forms ω_i are globally exact, i.e., there exist k functions f_i and k 1-forms α_i such that*

$$\eta_i=df_i,\quad \omega_i=d\alpha_i,$$

for all $1\leqslant i\leqslant k$, and with $(\alpha_i)_{|V}=0$ and $\alpha_i(\xi_j)=0$. Let $\overline{C_1},\ldots,\overline{C_k}$ be the vector fields on M, defined by

$$\iota_{\overline{C_i}}\eta_j=0,\quad \iota_{\overline{C_i}}\omega_j=\delta_{ij}\alpha_j,$$

for all $1\leqslant i,j\leqslant k$, and define

$$C_i=f_i\xi_i+\overline{C_i},$$

for $1\leqslant i\leqslant k$, where $\xi_1,\ldots,\xi_k$ are the Reeb vector fields. Then

$$\iota_{C_i}\eta_j=\delta_{ij}f_j,\quad \iota_{C_i}\omega_j=\delta_{ij}\alpha_j.$$

If the vector fields $C_1,\ldots,C_k$ satisfy (i)–(ii) *then there exists a unique vector bundle structure on M that is isomorphic to the stable bundle of k^1-covelocities $\mathbf{R}^k\times(T_k^1)^*S$ of the singular submanifold S of $C=C_1+\cdots+C_k$. Moreover, this isomorphism transports the canonical k-cosymplectic structure and the canonical vector field of $\mathbf{R}^k\times(T_k^1)^*S$ to (η_i,ω_i,V) and C, respectively.*

Proof: The vector field $C=C_1+\cdots+C_k$ satisfies the conditions (i)–(iv) in Theorem III.9, and then there exists a vector bundle isomorphism ϕ over the manifold S of the singular points of C:

$$\begin{tikzcd} N(S) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & M \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

where π is the canonical projection and π' is the projection induced from π via ϕ. Then, for each

$x\in S$, we can transport the k-cosymplectic structure on T_xM to a k-cosymplectic structure $(\eta_i(x),\omega_i(x),M_x)$ on the vector space $T_xM\oplus M_x$, with $M_x=\pi^{-1}(x)$, by using the linear isomorphism:

$$1+\phi_x: T_xM=T_xS\oplus N_xS\to T_xS\oplus M_x.$$

In fact, if we put $f_i(x)=\phi_x(\xi_i(x))$ and $W_i(x)=\phi_x(V_i(x))$, we have

$$M_x=\langle f_1(x),\ldots,f_k(x)\rangle\oplus W_1(x)\oplus\cdots\oplus W_k(x),$$

and we define a linear isomorphism,

$$G_x: M_x\to\mathbf{R}^k\times(T_x^*S\oplus\cdots\oplus T_x^*S),$$

by

$$G_x(f_i,0,...,0)=(e_i0,...,0),$$

$$G_x(0,0,...,v_i,\ldots,0)=(0,0,...,\iota_{v_i}\omega_i,\ldots,0),$$

where $\{e_i,1\leqslant i\leqslant k\}$ is the canonical basis of $\mathbf{R}^k$ and $v_i\in W_i(x)$. We finally get a vector bundle isomorphism:

$$G: M=\cup_{x\in S}M_x\to\mathbf{R}^k\times(T_k^1)^*S=\cup_{x\in S}(\mathbf{R}^k\times(T_x^*S\oplus\cdots\oplus T_x^*S)).$$

■

IV. HAMILTONIAN SYSTEMS ON *k*-COSYMPLECTIC MANIFOLDS

In order to introduce dynamics on a k-cosymplectic manifold we need to consider a Hamiltonian function defined on it. The dynamics will be given by k-vector fields, thus we first recall this notion, which is a natural extension of the notion of vector fields. Let M be an arbitrary manifold and

$$\tau^k: T_k^1M\to M,$$

its tangent bundle of k^1-velocities. Let us recall that T_k^1M is the manifold of 1-jets at $0\in\mathbf{R}^k$ of mappings from R^k into M.

Definition IV.1: A section $s: M\to T_k^1M$ of the projection τ^k will be called a *k-vector field* on M.

Since T_k^1M may be canonically identified with the Whitney sum of k copies of TM, say

$$T_k^1M\equiv TM\oplus\cdots\oplus TM,$$

we deduce that a k-vector field s defines k vector fields $X_1,\ldots,X_k$ on M by projecting s onto every factor. From now on, we shall identify s with the k-tuple $(X_1\ldots,X_k)$.

Definition IV.2: An application $\sigma: U\subset\mathbf{R}^k\to M$ defined on some open neighborhood of 0 $\in\mathbf{R}^k$ will be called an *integral section* of a k-vector field $(X_1,\ldots,X_k)$ passing through a point $x\in M$ if and only if

$$\sigma(0)=x,\quad \sigma_*(t)\left(\frac{\partial}{\partial t^i}\right)=X_i(\sigma(t))\quad \text{for all}\quad t\in U.$$

We say that a k-vector field $(X_1,\ldots,X_k)$ on M is *integrable* if there is an integral section passing through each point of M.

Let us remark that if σ is an integral section of a k-vector field $(X_1,\ldots,X_k)$, then each curve on M defined by $\sigma_i=\sigma\circ J_i$, where $J_i:\mathbf{R}\to\mathbf{R}^k$ is the natural inclusion $J_i(t)=(0,...,t,\ldots,0)$, is an integral curve of the vector field X_i on M, with $1\leqslant i\leqslant k$.

Now, assume that M is a k-cosymplectic manifold with k-cosymplectic structure $(\eta_1,\ldots,\eta_k,\omega_1,\ldots,\omega_k,V)$. We define two vector bundle morphisms $\Omega^\flat$ and $\Omega^\sharp$ as follows:

$$\Omega^{\flat}:TM \to (T_k^1)^*M$$

$$X \to \Omega^{\flat}(X) = (\iota_X \omega_1 + \eta_1(X)\,\eta_1, \ldots, \iota_X \omega_k + \eta_k(X)\,\eta_k)$$

and

$$\Omega^{\sharp}:T_k^1 M \to T^*M$$

$$(X_1, \ldots, X_k) \to \Omega^{\sharp}(X_1, \ldots, X_k),$$

such that

$$\Omega^{\sharp}(X_1, \ldots, X_k)(Y) = \mathrm{trace}((\Omega^{\flat}(X_j))_i(Y)) = \sum_{i=1}^{k} (\Omega^{\flat}(X_i))_i(Y) = \sum_{i=1}^{k} (\omega_i(X_i, Y) + \eta_i(X_i)\,\eta_i(Y)),$$

for all $Y \in TM$. The above morphisms induce two morphism of $C^{\infty}(M)$ modules between the corresponding spaces of sections.

Remark IV.3: If $k=1$ then $\Omega^{\flat} = \Omega^{\sharp}$ is defined from TM onto T^*M, and it is the morphism $\chi_{\eta,\omega}$ defined on the cosymplectic manifold (M, η, ω) by (see Refs. 30 and 31)

$$\chi_{\eta,\omega}(X) = \iota_X \omega + \eta(X)\,\eta.$$

If $(s^i, x^{\alpha}, x^i_{\alpha};\ 1 \leqslant i \leqslant k,\ 1 \leqslant \alpha \leqslant n)$ are Darboux coordinates for the k-cosymplectic structure (η_i, ω_i, V), and if the k-vector field $(X_1, \ldots, X_k)$ is expressed with respect to this system by

$$X_i = (X_i)^j \frac{\partial}{\partial s^j} + (X_i)^{\alpha} \frac{\partial}{\partial x^{\alpha}} + (X_i)^j_{\alpha} \frac{\partial}{\partial x^i_{\alpha}},$$

then

$$\Omega^{\sharp}(X_1, \ldots, X_k) = \sum_i (X_i)^i \, ds^i - \sum_{i,\alpha} (X_i)^i_{\alpha} \, dx^{\alpha} + \sum_{i,\alpha} (X_i)^{\alpha} \, dx^i_{\alpha}. \tag{4}$$

Let $H:M \to \mathbf{R}$ be a function on M. If $(X_1, \ldots, X_k)$ is a k-vector field on M, the equations

$$\eta_i(X_j) = \delta_{ij}, \quad \forall i, j,$$

$$\Omega^{\sharp}(X_1, \ldots, X_k) = dH + \sum_{i=1}^{k} (1 - \xi_i(H))\,\eta_i, \tag{5}$$

would imply

$$(X_i)^j = \delta_{ij} \quad \frac{\partial H}{\partial x^{\alpha}} = -\sum_{i=1}^{k} (X_i)^i_{\alpha}, \quad \frac{\partial H}{\partial x^i_{\alpha}} = (X_i)^{\alpha}. \tag{6}$$

From these local conditions we can define, in a neighborhood of each point $x \in M$, a k-vector field that satisfies (6). For example, we can put

$$(X_i)^j = \delta_{ij}, \quad (X_1)^1_{\alpha} = \frac{\partial H}{\partial x^{\alpha}}, \quad (X_i)^j_{\alpha} = 0, \quad \text{for } i \neq 1 \neq j, \quad (X_i)^{\alpha} = \frac{\partial H}{\partial x^i_{\alpha}}.$$

Now one can construct a global k-vector field, which is a solution of (6), by using a partition of the unity.

Remark IV.4: Equations (6) have not, in general, a unique solution. In fact, if we denote by $\mathscr{M}_k(C^{\infty}(M))$ the space of matrices of order k whose entries are functions on M, and we define the map

$$\eta^{\sharp}: T^1_k M \to \mathscr{M}_k(C^\infty(M))$$

$$(X_1,\ldots,X_k) \to (\eta_i(X_j)),$$

the solutions of (6) are given by $(X_1,\ldots,X_k)+(\ker\,\Omega^{\sharp}\cap\ker\,\eta^{\sharp})$, where $(X_1,\ldots,X_k)$ is a particular solution.

Definition IV.5: Any k-vector field $(X_1,\ldots,X_k)$ on M, such that

$$\eta_i(X_j)=\delta_{ij},$$

$$\Omega^{\sharp}(X_1,\ldots,X_k)=dH+\sum_{i=1}^{k}(1-\xi_i(H))\eta_i,$$

for all $1\leqslant i,j\leqslant k$, will be called an *evolution k-vector field* on M associated with the Hamiltonian function H.

Let $(X_1,\ldots,X_k)$ be an evolution k-vector field associated to H, and assume that it is integrable. Let

$$\sigma: \mathbf{R}^k \to M$$

$$(t^i) \to (\sigma^j(t^i), \alpha^\alpha(t^i), \sigma^j_\alpha(t^i)),$$

be an integral section of $(X_1,\ldots,X_k)$; then, we have

$$\frac{\partial\sigma^j}{\partial t^i}=\delta_{ij},\quad \frac{\partial\sigma^\alpha}{\partial t^i}=(X_i)^\alpha,\quad \frac{\partial\sigma^j_\alpha}{\partial t^i}=(X_i)^j_\alpha,$$

for all $1\leqslant i,\ j\leqslant k$ and $1\leqslant\alpha\leqslant n$. Therefore, Eqs. (6) give

$$\frac{\partial H}{\partial x^\alpha}=-\sum_{i=1}^{k}\frac{\partial\sigma^i_\alpha}{\partial t^i},$$

$$\frac{\partial H}{\partial x^i_\alpha}=\frac{\partial\sigma^\alpha}{\partial t^i},$$

with $1\leqslant i\leqslant k$ and $1\leqslant\alpha\leqslant n$, which are the field equations for H.

Remark IV.6: Let $(X_1,\ldots,X_k)$ be an evolution k-vector field. Since $\eta_i(X_j)=\delta_{ij}$, it follows that the vector fields $X_1,\ldots,X_k$ on M are linearly independent.

Example IV.7: We shall use the above formalism to obtain an intrinsic version for the electrostatic equations. Consider $\mathbf{R}^3$ with a metric g with components g_{ij}. Let $\phi:\mathbf{R}^3\to\mathbf{R}$ be the electric potential and $p=(p_1,p_2,p_3)$: $\mathbf{R}^3\to\mathbf{R}^3$ the electric field. Denote by (t^1,t^2,t^3) the standard coordinates on $\mathbf{R}^3$, and set $\sqrt{g}=\sqrt{\det g_{ij}}$. By $r(t)$ we denote the scalar function that gives the density of electric charge on $\mathbf{R}^3$. Consider on $M=\mathbf{R}^3\times(T^1_3)^*\mathbf{R}$ the canonical 3-cosymplectic structure $(\eta_i,\omega_i,V;1\leqslant i\leqslant 3)$. We denote by $(t^1,t^2,t^3,x,x^1,x^2,x^3)$ the local coordinates on $\mathbf{R}^3\times(T^1_3)^*\mathbf{R}$ induced by the standard coordinate (x) on $\mathbf{R}$, and define a Hamiltonian function,

$$H:\mathbf{R}^3\times(T^1_3)^*\mathbf{R}\to\mathbf{R}$$

$$(t^1,t^2,t^3,x,x^1,x^2,x^3)\to\sqrt{g}\left(4\pi r(t^1,t^2,t^3)x+\frac{1}{2g}\,g_{ij}x^ix^j\right).$$

Consider the equations

$$\eta i_i(X_j)=\delta_{ij} \quad (1\leqslant i,j\leqslant 3),$$

$$\Omega^\sharp(X_1,X_2,X_3)=dH+\sum_{i=1}^{3}(1-\xi_i(H))\eta_i, \tag{7}$$

where ξ_1, ξ_2, ξ_3 are the Reeb vector fields and (X_1,X_2,X_3) is a 3-vector field on $\mathbf{R}^3\times(T_3^1)^*\mathbf{R}$. Let $\sigma:\mathbf{R}^3\to\mathbf{R}^3\times(T_3^1)^*\mathbf{R}$, $\sigma(t)=(t,\psi(t),\psi^1(t),\psi^2(t),\psi^3(t))$ be an integral section of an evolution 3-vector field that is a solution of (7). Then we obtain

$$4\pi r(t^1,t^2,t^3)\sqrt{g}=-\left(\frac{\partial\psi^1}{\partial t^1}+\frac{\partial\psi^2}{\partial t^2}+\frac{\partial\psi^3}{\partial t^3}\right),$$

$$\frac{1}{\sqrt{g}}\left(\sum_{i=1}^{3}g_{ij}\psi^j\right)=\frac{\partial\psi}{\partial t^i},$$

which are the electrostatic equations, and then the components $\psi(t)$ and $(\psi^1(t),\psi^2(t),\psi^3(t))$ of σ are, respectively, the electric potential ϕ and electric field $p=(p_1,p_2,p_3)$ on $\mathbf{R}^3$.

Actually, in most of the real electrostatic problems, one has to deal with finite regions and with the boundary conditions established in the limits of those regions. To do that the Green identities have to be used.

Thus, the suitable boundary conditions to the Poisson (Laplace) equations to present a unique well-behaved solution inside a finite region are modeled. Physically, we know that the specification of the potential on a closed surface defines a unique boundary problem (Dirichlet boundary conditions). The specification of the electric field at any point of the closed surface, i.e., the specification of the distribution charge on the surface (Neumann boundary conditions), also defines a unique problem. Those propositions can be shown by taking into account the first Green identity (see Ref. 38 and the references therein).

V. R^k-VALUED HAMILTONIANS

We can also consider Hamiltonian functions $\mathbf{R}^k$-valued on a k-cosymplectic manifold M.

In Darboux coordinates we have

$$H=(H^1(s^i,x^\alpha,x^i_\alpha),\ldots,H^k(s^i,x^\alpha,x^i_\alpha)),$$

or equivalently,

$$H=H^ie_i,$$

where $\{e_1,\ldots,e_k\}$ denotes the standard basis in $\mathbf{R}^k$.

Definition V.1: If M is a k-cosymplectic manifold, a vector field X on M such that

$$\eta_i(X)=1, \quad \text{for all } i=1,...,k,$$

$$\Omega^b(X)=dH+\left[\sum_{i=1}^{k}((1-\xi_i(H^i))\eta_i)e_i\right], \tag{8}$$

where $\xi_1,\ldots,\xi_k$ are the Reeb vector fields associated with the k-cosymplectic structure, will be called a *k-Hamiltonian system* associated to H.

In Darboux coordinates, we have

$$X=\frac{\partial}{\partial s^1}+\cdots+\frac{\partial}{\partial s^k}+X^\alpha\frac{\partial}{\partial x^\alpha}+X^i_\alpha\frac{\partial}{\partial x^i_\alpha},$$

where

$$\frac{\partial H^i}{\partial s^j}=0, \quad \text{if } i\neq j, \quad \frac{\partial H^i}{\partial x^\alpha}=-X^i_\alpha, \quad \frac{\partial H^i}{\partial x^j_\alpha}=\delta_{ij}X^\alpha.$$

If

$$\sigma:(-\epsilon,\epsilon)\subset \mathbf{R}\rightarrow M$$

$$t\rightarrow(\sigma^i(t),\sigma^\alpha(t),\sigma^i_\alpha(t))$$

is an integral curve of X, then we obtain the Hamilton equations associated to the $\mathbf{R}^k$-valued Hamiltonian H:

$$\frac{\partial H^i}{\partial x^\alpha}=-\frac{d\sigma^i_\alpha}{dt},$$

$$\frac{\partial H^i}{\partial x^j_\alpha}=\delta_{ij}\frac{d\sigma^\alpha}{dt}.$$

From these equations and using the same argument as in Refs. 24 and 25, we can prove the following result.

Proposition V.2: An admissible Hamiltonian function $H=(H^1,\ldots,H^k)$ *is necessarily of the form*

$$H^i=\sum_{\alpha=1}^{n}\varphi_\alpha(s^i,x_1,\ldots,x_n)x^i_\alpha+\psi^{\,i}(s^i,x_1,\ldots,x_n),$$

for $1\leqslant i\leqslant k$.

Given an admissible Hamiltonian function $H=(H^1,\ldots,H^k)$ on M, we can define the local vector field,

$$X=\frac{\partial}{\partial s^1}+\cdots+\frac{\partial}{\partial s^k}+\frac{\partial H^i}{\partial x^i_\alpha}\frac{\partial}{\partial x^\alpha}-\frac{\partial H^i}{\partial x^\alpha}\frac{\partial}{\partial x^i_\alpha},$$

which satisfies (8). By using a partition of the unity, it is possible to define a unique global vector field X on M satisfying (8).

An application. Let $L:\mathbf{R}\times TQ\rightarrow\mathbf{R}$ be a hyperregular Lagrangian function, that is, the Legendre transformation Leg:$\mathbf{R}\times TQ\rightarrow\mathbf{R}\times T^*Q$ defined by L is a diffeomorphism. Denote by $H=E_L\circ\mathrm{Leg}^{-1}$ the Hamiltonian energy, where $E_L=C(L)-L$, C being the infinitesimal generator of the dilations on TQ. We construct a 2-cosymplectic structure on the manifold $M=\mathbf{R}^2\times(TQ\oplus T^*Q)$ as follows.

First of all, let ω_Q denote the canonical symplectic form on T^*Q and $\omega_L=\mathrm{Leg}^*\,\omega_Q$ the Poincaré–Cartan two-form on TQ.[39] If (q^α) are coordinates on Q we denote by (q^α,v^α) and (q^α,p_α) the induced coordinates on TQ and T^*Q, respectively. So, we have coordinates $(t^1,t^2,q^\alpha,v^\alpha,p_\alpha)$ on M. Denote by τ: $M\rightarrow\mathbf{R}\times TQ$ and $\pi:M\rightarrow\mathbf{R}\times T^*Q$ the canonical projections, locally given by

$$\tau(t^1,t^2,q^\alpha,v^\alpha,p_\alpha)=(t^1,q^\alpha,v^\alpha),$$

$$\pi(t^1,t^2,q^\alpha,v^\alpha,p_\alpha)=(t^2,q^\alpha,p_\alpha).$$

If we put

$$\eta_1=dt^1, \quad \eta_2=dt^2, \quad \omega_1=\tau^*\omega_L, \quad \omega_2=\pi^*\omega_Q,$$

we deduce that $(\eta_1,\eta_2,\omega_1,\omega_2,V)$ is a 2-cosymplectic structure on M where $V=\mathrm{Ker}\,T\rho$, $\rho:M\rightarrow Q$ being the canonical projection.

Consider now a Hamiltonian function,

$$H:M\to\mathbf{R}^2,$$

given by

$$H(t^1,t^2,X_q,\alpha_q)=(\tau^*E_L,\langle X_q,\alpha_q\rangle-\tau^*L).$$

In local coordinates we obtain

$$H(t^1,t^2,q^\alpha,v^\alpha,p_\alpha)=\left(v^\alpha\frac{\partial L}{\partial v^\alpha}-L,q^\alpha p_\alpha-L\right).$$

Let now X be a vector field on M, which is Hamiltonian for H. After a direct computation, we deduce

$$X=\frac{\partial}{\partial t^1}+\frac{\partial}{\partial t^2}+v^\alpha\frac{\partial}{\partial q^\alpha}+X^1_\alpha\frac{\partial}{\partial v^\alpha}-\frac{\partial H}{\partial q^\alpha}\frac{\partial}{\partial p_\alpha},$$

where

$$\frac{\partial^2 L}{\partial t^1\,\partial v^\beta}+v^\alpha\frac{\partial^2 L}{\partial q^\alpha\,\partial v^\beta}+X^1_\alpha\frac{\partial^2 L}{\partial v^\alpha\,\partial v^\beta}-\frac{\partial L}{\partial q^\beta}=0.$$

If $\sigma(t)=(\sigma_1(t),\sigma_2(t),\sigma^\alpha(t),\sigma^1_\alpha(t),\sigma^2_\alpha(t))$ is an integral curve of X, we have

$$\frac{d\sigma^\alpha}{dt}=v^\alpha,\quad \frac{d\sigma^1_\alpha}{dt}=X^1_\alpha,\quad \frac{d\sigma^2_\alpha}{dt}=-\frac{\partial H}{\partial q^\alpha}.$$

Therefore, the projections of σ onto Q satisfy simultaneously the Euler–Lagrange and Hamilton equations.

VI. MULTISYMPLECTIC FORMALISM ON JET BUNDLES

In this section we obtain the Hamiltonian formalism above as a particular case of the multi-symplectic formalism on jet bundles.[12]

Let $\pi:E\to M$ be a fibered manifold. We denote by Λ^k_0E and Λ^k_1E the following vector bundles on E:

$$\Lambda^k_0E=\{\sigma\in\Lambda^kE/\iota_v\sigma=0,\quad \forall_v\in V\pi\},$$

$$\Lambda^k_1E=\{\sigma\in\Lambda^kE/\iota_{v_1}\iota_{v_2}\sigma=0,\quad \forall v_1,v_2\in V\pi\},$$

where $V\pi$ is the vertical bundle of π. Here, Λ^rE denotes the space of r-forms on the manifold E, that is,

$$\Lambda^rE=\cup_{z\in E}\Lambda^r(T^*_zE).$$

If we take fibered coordinates (s^i,x^α), then each element in Λ^k_0E can be expressed as $x(s^i,x^\alpha)d^ks$, and each element of Λ^k_1E as $x(s^i,x^\alpha)d^ks+x^i_\alpha(s^i,x^\alpha)dx^\alpha\wedge d^{k-1}s_i$, where $d^ks=ds^1\wedge\cdots\wedge ds^k$ and $d^{k-1}s^i=\iota(\partial/\partial s^i)\ d^ks$. Thus, we can introduce local coordinates (s^i,x^α,x) and $(s^i,x^\alpha,x,x^i_\alpha)$ on Λ^k_0E and Λ^k_1E, respectively.

We denote by $\mathscr{M}\pi=\Lambda^k_1E$. Since Λ^k_0E is a vector subundle of $\mathscr{M}\pi$, then $\mathscr{M}\pi/\Lambda^k_0E=J^1\pi^*$ is a quotient manifold of dimension $(k+1)n+k$ with local coordinates $(s^i,x^\alpha,x^i_\alpha)$.

Moreover, we have the following canonical projections:

$$\mu:\mathscr{M}\pi\to J^1\pi^*,$$

$$\pi^*_{1,0}:J^1\pi^*\to E,$$

$$\pi^*_1:J^1\pi^*\to M.$$

On $\mathscr{M}\pi$ there is a canonical k-form Θ, which is defined as follows:

$$\Theta_\omega(X_1,\ldots,X_k)=\omega(\nu_*X_1,\ldots,\nu_*X_k),$$

for all $X_i\in T_\omega\mathscr{M}\pi$ and $\omega\in\mathscr{M}\pi$, where $\nu:\mathscr{M}\pi\to E$ is the canonical projection.

Definition VI.1: The $(k+1)$-form on $\mathscr{M}\pi$, defined by

$$\Omega=d\Theta,$$

is called the *multisymplectic structure* on $\mathscr{M}\pi$.

Definition VI.2: A global section h: $J^1\pi^*\to\mathscr{M}\pi$ of μ is called a *Hamiltonian*. Given a Hamiltonian h on $J^1\pi^*$, we can define a $(k+1)$-form on $J^1\pi^*$ given by

$$\Omega_h=-h^*\Omega.$$

Definition VI.3: A section $\gamma:M\to J^1\pi^*$ of π^*_1 satisfies the Hamilton equations associated to a Hamiltonian h if and only if

$$\gamma^*(\iota_X\Omega_h)=0,$$

for all vector field X on $J^1\pi^*$.

Put

$$h(s^i,x^\alpha,x^i_\alpha)=(s^i,x^\alpha,x=-H(s^i,x^\alpha,x^i_\alpha),x^i_\alpha),$$

with $H:J^1\pi^*\to\mathbf{R}$.

Then a section $\gamma(s^i)=(s^i,\gamma^\alpha(s^i),\gamma^j_\alpha(s^i))$ as in Definition VI.3 satisfies the following system of differential equations:

$$\frac{\partial H}{\partial x^\alpha}=-\sum_{i=1}^k\frac{\partial\gamma^i_\alpha}{\partial s^i},$$

$$\frac{\partial H}{\partial x^i_\alpha}=\frac{\partial\gamma^\alpha}{\partial s^i},$$

which are the field equations for H (see Sec. IV).

Next, we shall consider the particular case where $\pi:E\to M$ is the trivial bundle $(\mathbf{R}^k\times Q,\pi,\mathbf{R}^k)$.

In this case we get

$$\mathscr{M}\pi=\mathbf{R}^k\times\mathbf{R}\times(T^1_k)^*Q,\quad J^1\pi^*=\mathbf{R}^k\times(T^1_k)^*Q.$$

Consider the canonical k-cosymplectic structure $((\eta_0)_i,(\omega_0)_i,V_0)$ on $J^1\pi^*=\mathbf{R}^k\times(T^1_k)^*Q$ described in Sec. II. Notice that $(\omega_0)_i=-d(\lambda_0)_i$, where $(\lambda_0)_i=x^i_\alpha\,dx^\alpha$, and $(t^i,x^\alpha,x^i_\alpha)$ are Darboux coordinates (see Sec. II).

If $h:J^1\pi^*=\mathbf{R}^k\times(T^1_k)^*Q\to\mathscr{M}=\mathbf{R}^k\times\mathbf{R}\times(T^1_k)^*Q$ is a Hamiltonian on $J^1\pi^*$ such that

$$h(t^i,x^\alpha,x_{\ \alpha})=(t^i,-H(t^i,x^\alpha x^i_\alpha),x^\alpha,x^\alpha_i),$$

then we obtain the following.

Lemma VI.4: We have

$$\Theta_h = -H(\eta_0)_1 \wedge \cdots \wedge (\eta_0)_k + \sum_{i=1}^{k} (\lambda_0)_i \wedge \iota_{\xi_i}((\eta_0)_1 \wedge \cdots \wedge (\eta_0)_k),$$

where $(\xi_0)_i = \partial/\partial t^i (1 \leq i \leq k)$, *are the Reeb vector fields on* $\mathbf{R}^k \times (T_k^1)^* Q$.

Thus, we deduce the following.

Proposition VI.5: A section ψ *of the canonical projection* $\pi^*: \mathbf{R}^k \times (T_k^1)^* Q \to \mathbf{R}^k$ *satisfies the Hamilton equations in Definition VI.3 if and only if* ψ *is an integral section of an evolution k-vector field on* $\mathbf{R}^k \times (T_k^1)^* Q$ *associated to* H.

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